A Time-Domain Vector Potential Formulation for the Solution of Electromagnetic Problems

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***Abstract*—We present an alternative vector potential formulation of Maxwell's equations derived upon introduction of a quantity related to the Hertz potential. Once space and time are discretized, within this formulation the electric field and vector potential components are condensed in the same point in the elementary cell. In three dimensions the formulation offers an alternative to finite-difference time-domain (FDTD) method; when reduced to a two-dimensional (2-D) problem, only two variables, instead of three, are necessary, implying a net memory saving of 1/3 with respect to FDTD.**

I. Introduction

SEVERAL numerical techniques have been devised in the past to solve electromagnetic problems in the time domain. Much work has also been devoted to the formulation of a condensed node representation of the field components in the elementary computational cell. Condensed node transmission-line matrix (TLM) [1], for example, achieved this goal at the price of a higher number of variables per elementary cell and consequently a larger memory requirement. The classical vector potential formulation, upon introduction of the scalar potential [2], ensures the condensed node representation of the field components, and it offers also the advantage that each field component can be propagated in time and space independently from the others. The disadvantage with this approach is mainly the difficulty in imposing proper boundary conditions, since there is no access to the field components.

In the vector potential approach presented here, the introduction of a new vector quantity $(\boldsymbol{K})$ and its relation with the electric field guarantees the advantages of the classical vector potential formulation, such as the condensed node representation of the fields components. In addition, this formulation gives direct access to the electric field so that boundary conditions can be easily implemented, and the equations are all first order in time. A net reduction of 30% in memory requirement is achieved when solving a two-dimensional (2-D) problem. Compared to other work on modified finite-difference time-domain (FDTD) formulations obtaining reduction in memory requirements [3], our approach is not limited to specific structures.

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II. Vector Potential Formulation

Consider a charge-free medium, where both the permeability and permittivity are frequency independent but they can vary in space. It is possible to rewrite Maxwell's equations upon introduction of the vector potential $\boldsymbol{A}$ in the following form [4]:

$$\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\varepsilon}\nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{A}\right) \tag{1a}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} = -\boldsymbol{E} - \nabla\phi \tag{1b}$$

$$\varepsilon\mu\frac{\partial \phi}{\partial t} = -\frac{1}{\varepsilon}\nabla\cdot(\varepsilon\boldsymbol{A}) \tag{1c}$$

where (1c) is the generalized Lorentz–Gauge condition for inhomogeneous media; it reduces to the standard Lorentz–Gauge condition when the dielectric constant is space invariant. We take a different approach in solving (1) by introducing a new vector quantity $\boldsymbol{K}_e$, such that

$$\frac{\partial \boldsymbol{K}_e}{\partial t} = -\boldsymbol{E}. \tag{2}$$

This allows us to rewrite system (1) exploiting a new set of variables which is more convenient from the computational point of view. One important property of the vector $\boldsymbol{K}_e$ can be derived by comparing the curl of (1b) and the curl of (2):

$$\nabla\times\boldsymbol{A} = \nabla\times\boldsymbol{K}_e. \tag{3}$$

Hence it is possible to rewrite system (1) without the use of the scalar potential, which is somehow included within $\boldsymbol{K}_e$. We can exploit this property since in the calculation of the electric field only the curl of $\boldsymbol{A}$ is needed and not $\boldsymbol{A}$ itself [5]–[7]:

$$\frac{\partial \boldsymbol{E}}{\partial t} = \frac{1}{\varepsilon}\nabla \times \left(\frac{1}{\mu}\nabla \times \boldsymbol{K}_e\right) \tag{4a}$$

$$\frac{\partial \boldsymbol{K}_e}{\partial t} = -\boldsymbol{E}. \tag{4b}$$

In a totally analogous way we can rewrite Maxwell's equations upon introduction of the magnetic vector potential $\boldsymbol{F}$. The equations are further simplified by using $\boldsymbol{K}_h$, defined as $(\partial\boldsymbol{K}_h/\partial t) = \boldsymbol{H}$, so that we obtain

$$\frac{\partial \boldsymbol{H}}{\partial t} = \frac{1}{\mu}\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \boldsymbol{K}_h\right) \tag{5a}$$

$$\frac{\partial \boldsymbol{K}_h}{\partial t} = \boldsymbol{H}. \tag{5b}$$

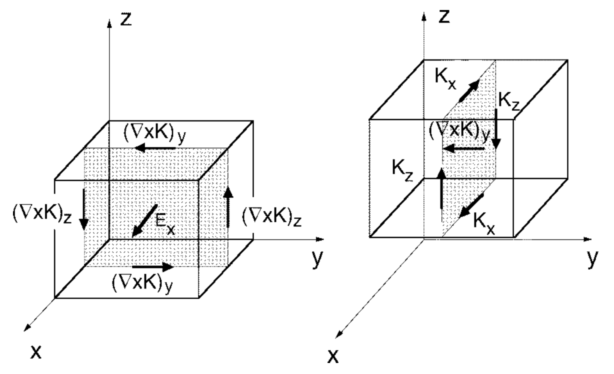

Fig. 1. (a) Along the sides of the elementary cell lie the components of the vector $(\nabla \times \boldsymbol{K})$, from which E_x can be calculated. (b) The curl of $\boldsymbol{K}$ is calculated from its components.

Systems (4) and (5) are totally equivalent, and one may use either one depending on which one is more convenient, given the symmetry and the nature of the electromagnetic problem. Notice that the new vector quantities introduced $(\boldsymbol{K}_e, \boldsymbol{K}_h)$ can be easily related to the Hertz potentials (Π_e, Π_h). In fact, combining the equation defining the magnetic Hertz potential [8]

$$\boldsymbol{E} = \mu \frac{\partial}{\partial t}(\nabla \times \Pi_h) \tag{6}$$

together with (2), one obtains

$$\frac{\partial K_e}{\partial t} = -\mu \frac{\partial}{\partial t}(\nabla \times \Pi_h) \tag{7}$$

which, upon simple integration, leads to

$$K_e = -\mu \nabla \times \Pi_h. \tag{8}$$

Similarly, it is possible to derive the corresponding equation for the electric Hertz potential.

Let us now focus on system (4) and show how all the field components can be thought to belong to the same point in the elementary computational cell, which is the basis for the condensed node field representation. Equation (4b) ensures that the vectors $\boldsymbol{E}$ and $\boldsymbol{K}_e$ are parallel. Furthermore, the spatial derivatives described in (4a) are to be discretized in space using the central difference scheme, which ensures the same location for the components of $\boldsymbol{E}$ and $\boldsymbol{K}_e$ in the unit cell. As an example we show that E_x and K_{ex} belong to the same location in the elementary cell. The same is true for all the other pairs of components.

Consider the elementary cell as shown in Fig. 1(a). Begin with the assumption that the x-component of the electric field is located in the center of the cell. According to (4a) the time derivative of its value can be evaluated as the curl of the vector $(\nabla \times \boldsymbol{K}_e)$, whose components lie along the sides of the elementary cell, as visible in Fig. 1(a). Focus now on the components $(\nabla \times \boldsymbol{K}_e)_y$ visible in Fig. 1(b) and corresponding to the top of the cell in Fig. 1(a). Fig. 1(b) shows how to calculate the curl of $\boldsymbol{K}_e$ from its components. By comparing Fig. 1(a) and (b), it is clear that E_x and K_{ex} belong to the same point in space. Clearly this is not a formal point, because not only are E_x and K_{ex} located in the center of the cell, but also the other components of $\boldsymbol{K}$ and $\boldsymbol{E}$ as well. The condensed node representation of the electric field shows, as an example, in the implementation of metal boundary conditions. In fact the interface is located exactly in the center of the computational cell, and the metal treatment can be obtained by simply setting to zero the field component at that location. On the contrary in the standard FDTD, separate treatment for each field component must be implemented at the same cell.

As in FDTD, the stability condition for this numerical technique, is set by the Courant condition [2], which requires that in one time step the propagating wave must not travel through more than one computatinal cell.

III. Formulation for the TM and TE Cases

Let us consider the TM electromagnetic problem where only E_z, H_x, and H_y field components are present. Due to the peculiar symmetry, this case can be addressed as a 2-D problem. Within this vector potential formulation, it is not necessary to use H_x and H_y, and we only need be concerned with E_z. The presence of E_z implies the existence of K_{ez} from (2). As a consequence, (4) is projected along the z-direction giving

$$\begin{aligned} \frac{\partial E_z}{\partial t} &= \frac{1}{\varepsilon}\nabla_{x,y} \times \left(\frac{1}{\mu}\nabla_{x,y} \times (K_e)_z\right) \\ \frac{\partial (K_e)_z}{\partial t} &= -E_z \end{aligned} \tag{9}$$

where $\nabla_{x,y}$ is $(\partial/\partial x)\hat{x} + (\partial/\partial y)\hat{y}$. This is not a formal point, since it causes a reduction in memory storage requirement of 1/3 with respect to the FDTD formulation where all three components $({}_E z, H_x$, and $H_y)$ must be treated. However, the computational speed is the same as in the FDTD algorithm, because at each time step the number of computer operations is the same.

We present here numerical experimental results for the internal electric field of a multilayer circular dielectric cylinder, assumed to be infinite in the z-direction. The incident radiation is a TM wave with respect to the cylinder symmetry axis. Again the symmetry is such that the problem can be solved in two dimensions. The three layers of the cylinder are shown in Fig. 2. The inner core has radius of 0.1 m and relative dielectric constant of $\varepsilon = 4.0$; the next two shells have radii of 0.15 and 0.2 m, respectively, and dielectric constants $\varepsilon = 3$ and $\varepsilon = 2$. The computational domain consists of 400 × 400 square cells with lateral size 0.5 mm, while the time resolution is 0.5 ps. The source of the radiation investing the cylinder is a plane wave for the z-component of the electric field at 1.5 GHz. The electric field (E_z) inside the cylinder, along its diameter perpendicular to the incident wave, is shown in Fig. 2 and compared to the exact solution [9].

Now let us consider the complementary TE case where only H_z, E_x, and E_y field components are present. It is convenient, here, to exploit the H, K_h formulation expressed by system (5), which can be further simplified since only the z-component of the magnetic field is present

$$\begin{aligned} \frac{\partial H_z}{\partial t} &= \frac{1}{\mu}\nabla_{x,y} \times \left(\frac{1}{\varepsilon}\nabla_{x,y} \times (K_h)_z\right) \\ \frac{\partial (K_h)_z}{\partial t} &= H_z. \end{aligned} \tag{10}$$

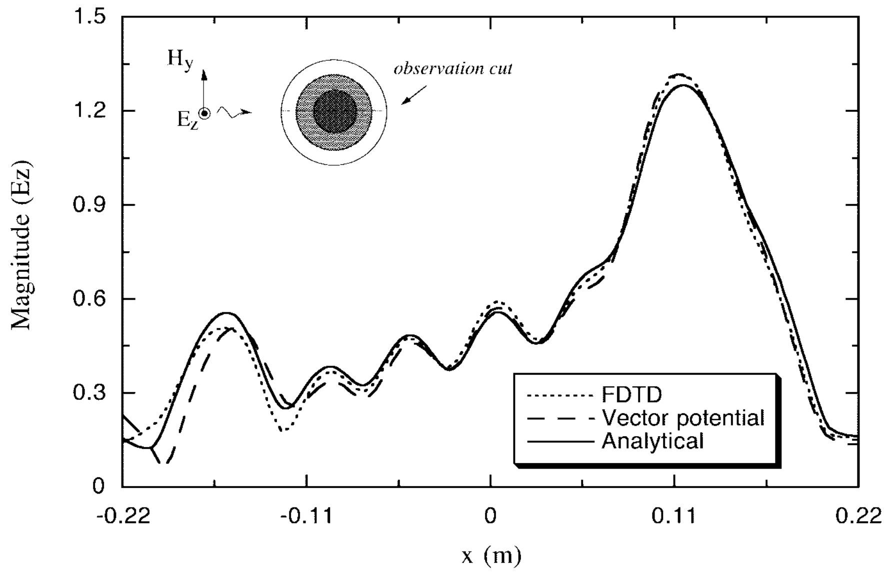

Fig. 2. Values of the magnitude of E_z inside the dielectric multilayer cylinder along the diameter, obtained with the three methods.

Notice that in this case too, only two scalar quantities (H_z, K_z) are needed to solve the TE problem, with the same advantages explained before.

IV. Conclusions

We have derived an alternative formulation of the Maxwell's equations with the use of the vector potential, and the equations can be further simplified with the introduction of the vector $\boldsymbol{K}_{e,h}$. The two alternative derivations, employing either the pairs $(\boldsymbol{K}_e, \boldsymbol{E})$ or $(\boldsymbol{K}_h, \boldsymbol{H})$, are completely equivalent, and one is allowed to choose the most convenient formulation according to the symmetry of the problem. In three dimensions the formulation offers an alternative to the usual FDTD method, since the same number of variables is needed to solve the electromagnetic problem. However, with the present formulation the electric and vector potential fields are condensed in the same point in space, within the computational cell. When reduced to a 2-D problem, TM or TE case, only two variables, instead of three, are to be used, implying a net memory saving of 1/3 over the usual FDTD formulation.

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